Highly oscillating thin obstacles

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(joint work with Ki-Ahm Lee and Martin Strömqvist)

Obstacle Problems

Let Ω be a bounded, open and connected domain in $\mathbb{R}^n$.

Definition

For any given function $\psi \in \mathcal{C}^2(\Omega)$(or $\psi \in H^1(\Omega)$), known as the obstacle, and $g \in H^1(\Omega)$ satisfying the compatibility condition $\psi \leq g$ on $\partial\Omega$, we say u is the solution of the obstacle problem if u is the minimizer of the functional

$$J(v) = \int_\Omega |\nabla v|^2 + 2fvdx$$

where v is in $K = \{v \in H^1(\Omega) : v \geq \psi \text{ in } \Omega, \quad v - g \in H_0^1(\Omega)\}$.

Obstacle Problems

If the obstacle is only defined on (n-1)-dimensional manifold Γ then we call such problems Signorini problem or Thin obstacle problem.

Definition

For any given function $\psi \in \mathcal{C}^2(\Omega)$(or $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$), $g \in H^1(\Omega)$ satisfying the compatibility condition $\psi \leq g$ on $\partial\Omega$, we say u is the solution of the obstacle problem if u is the minimize of the functional

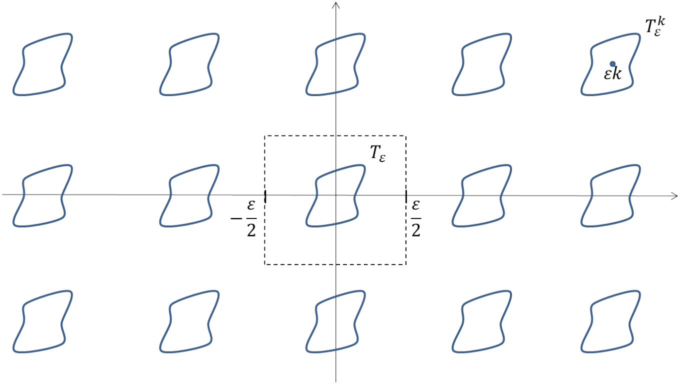
$$J(v) = \int_\Omega |\nabla v|^2 + 2fv dx$$

where v is in $K = \{v \in H^1(\Omega) : v \geq \psi \text{ on } (\Gamma \cap \Omega), \quad v - g \in H_0^1(\Omega)\}$.

In this talk, we assume that $n \geq 3$ and $g = 0$.

Periodical Background

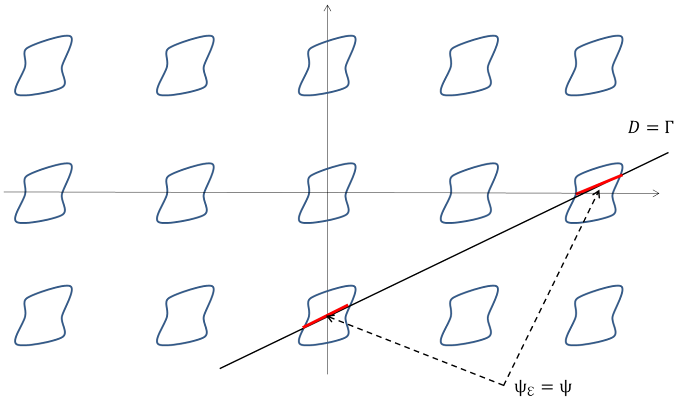
For a given open subset T of $B_{\frac{1}{2}}$, we define T_ε^k by $\varepsilon k + a_\varepsilon T$ and $\mathcal{T}_\varepsilon$ by $\cup T_\varepsilon^k$ where $k \in \mathbb{Z}^n$ and $a_\varepsilon < \varepsilon$.



Periodical Background

For any subset D of $\mathbb{R}^n$ and for any $\psi \in H^1(\mathbb{R}^n)$, we define $D_\varepsilon = D \cap T_\varepsilon$ and

$$\psi_\varepsilon = \psi \chi_{D_\varepsilon} = \begin{cases} \psi(x) & \text{if } x \in D_\varepsilon, \\ 0 & \text{if } x \notin D_\varepsilon \end{cases}$$



Highly Oscillating Obstacle Problem

Let u_ε be the solution of the obstacle problem with obstacle ψ_ε. More precisely, u_ε is the minimizer of the functional

$$J(v) = \int_\Omega |\nabla v|^2 + 2fvdx \qquad (M_\varepsilon)$$

for $v \in K_\varepsilon = \{v \in H^1(\Omega); v \geq \psi_\varepsilon \text{ in } \Omega, \quad v - g \in H^1_0(\Omega)\}$.
Then our main problem is to determine the limit u of the solution u_ε in terms of a limit equation it solves.

Definition

1. If $D = \Omega$, then we call (M_ε) a highly oscillating obstacle problem.
2. If $D = \Gamma$, $(n-1)$-dimensional manifold, then we call (M_ε) a highly oscillating thin obstacle problem.

Actually, the limit behavior of u_ε is related with the decay rate a_ε of holes.

Averaged Capacity

Definition

Let A be any compact subset of $\mathbb{R}^n$. Then, for any $n \geq 3$, the capacity of A is defined as follow

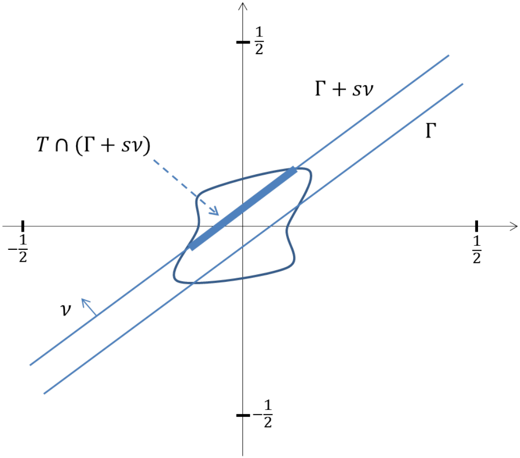
$$cap(A) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla \rho|^2 dx : \rho \in \mathcal{C}_0^\infty(\mathbb{R}^n), \rho \geq 1 \text{ on } A \right\}$$

Now we define the new concept of the capacity to describe the equation that satisfies the limit u of u_ε.
Let Γ be a hyper plane in $\mathbb{R}^n$ with normal $\nu \in S^{n-1}$. and let

$$\Gamma(s) = \Gamma_\nu(s) = \Gamma + s\nu.$$

be the family of hyper-planes.

Averaged Capacity



Averaged Capacity

Definition (Averaged capacity)

If $T \in \mathbb{R}^n$ and $\Gamma = \{(x - x_0) \cdot \nu = 0\}$,

$$f(s) = cap(T \cap \Gamma + s\nu)$$

is integrable, then we set

$$cap_\nu(T) = \int_{-\infty}^{\infty} f(s)ds$$

and call this quantity the averaged capacity of T with respect to ν.

Main Theroem

Theorem

Let $n \geq 3$ and u_ε be the solution of (M_ε) when $D = \Gamma = \{x \in \mathbb{R}^n : (x - x_0) \cdot \nu = 0\}$. We set we set $a_\varepsilon = \varepsilon^{\frac{n}{n-1}}$. Then $u_\varepsilon \to u$ weakly in $H_0^1(\Omega)$ and u is the unique minimizer of

$$J_\nu(v) = \int_\Omega |\nabla v|^2 + 2 f v dx + cap_\nu(T) \int_\Gamma \left((\psi - v)^+\right)^2 d\sigma, \quad v \geq 0 \qquad \text{(M)}$$

for a.e. $\nu \in S^{n-1}$.
In particular, if f is non-negative, then u is the solution of

$$-\triangle u + cap_\nu(T)(\psi - u)^+ d\sigma + f = 0.$$

Related Works

1. In 1997, Cioranescu and Murat proved that if $D = \Omega$ and $a_\varepsilon = \varepsilon^{\frac{n}{n-2}}$, then the solution u_ε of (M_ε) converges to u and u satisfies the following equation:

$$-\triangle u + \mu u = f \text{ in } \Omega$$

for some constant μ.

2. They also proved that if $D = \{x_n = 0\}$ and $a_\varepsilon = \varepsilon^{\frac{n-1}{n-2}}$, then the limit u satisfies

$$-\triangle u + \mu u d\delta_{\{x_n=0\}} = f \text{ in } \Omega.$$

3. In 2008, Caffarelli and Lee proved the similar result for the fully nonlinear operator F when $D = \Omega$. The decay rate a_ε is determined by the constant λ that makes

$$V(x) = |x|^{-\lambda}\Phi(\frac{x}{|x|})$$

a solution of $F(D^2V) = 0$ except the singular point.

4. In 2009, Caffarelli and Mellet proved the similar result when $T = T(k,\omega)$ is distributed randomly. If $cap(T(k,\omega))$ is stationary ergodic, $D = \Omega$, and $a_\varepsilon = \varepsilon^{\frac{n}{n-2}}$ then we have similar result to (1) and the limit doesn't depend on ω.

Corrector

Proposition (Existence of Corrector)

Let $\Gamma = \{x \in \mathbb{R}^n; (x - x_0) \cdot \nu = 0\}$ *and let* $a_\varepsilon = \varepsilon^{\frac{n}{n-1}}$. *Then, for almost all* $\nu \in S^{n-1}$, *there exists a sequence of functions* w_ε *satisfying*

1. $w_\varepsilon = 1$ *on* Γ_ε
2. $w_\varepsilon \to 0$ *weakly in* $H_0^1(\Omega)$
3. *For any sequence* $z_\varepsilon \in H_0^1(\Omega)$ *such that* $z_\varepsilon = 0$ *on* Γ_ε, $\|z_\varepsilon\|_{L^\infty(\Omega)} \leq C$ *and* $z_\varepsilon \to z$ *weakly in* $H_0^1(\Omega)$ *there holds*

$$\lim_{\varepsilon \to 0} < \triangle w_\varepsilon, \rho z_\varepsilon >_{H^{-1}, H_0^1} = cap_\nu(T) \int_\Gamma \rho z d\sigma \tag{2.1}$$

for all $\rho \in \mathcal{C}_0^\infty(\Omega)$.

Proof of Theorem

Lemma (Lower semicontinuity of the Energy)

Let $z_\varepsilon \in H_0^1$ be a sequence which is bounded uniformly on ε, $z_\varepsilon \to z$ weakly in H_0^1 and $z_\varepsilon = 0$ on Γ_ε. Then, we have

$$\liminf \int_\Omega |\nabla z_\varepsilon|^2 dx \geq \int_\Omega |\nabla z|^2 dx + cap_\nu(T) \int_\Gamma z^2 d\sigma.$$

The last term comes from the choice of the test function z_ε.

Sketch of proof.

From the proposition,

$$\int_\Omega |\nabla w_\varepsilon|^2 \rho^2 \to cap_\nu(T) \int_\Gamma \rho^2 d\sigma, \quad \rho \in \mathcal{C}_0^\infty(\Omega)$$

Apply this to $0 \leq \int_\Omega |\nabla(z_\varepsilon - w_\varepsilon \rho)|^2 dx$ and then apply $\rho = z_\varepsilon$. Then, we get the conclusion. □

Proof of Theorem

Lemma

Let u_ε be solution of (M_ε). Then we have

$$\liminf \int_\Omega |\nabla u_\varepsilon|^2 dx \geq \int_\Omega |\nabla u|^2 dx + cap_\nu(T) \int_\Gamma \left((\psi - u)^+\right)^2 d\sigma.$$

Proof.

We have the identity $u_\varepsilon = -(\psi - u_\varepsilon)^+ + (\psi - u_\varepsilon)^- + \psi$. Since $u_\varepsilon \geq \psi$ on Γ_ε, $(\psi - u_\varepsilon)^+ = 0$ on Γ_ε, we have

$$\liminf \int_\Omega |\nabla(\psi - u_\varepsilon)^+|^2 dx \geq \int_\Omega |\nabla(\psi - u)^+|^2 dx + cap_\nu(T) \int_\Gamma \left((\psi - u)^+\right)^2 d\sigma.$$

Then, the lemma follows from above and the general weak lower semi-continuity. □

Proof of Theorem

Lemma

Let u_ε be solution of (M_ε). Then we have

$$\limsup \int_\Omega |\nabla u_\varepsilon|^2 dx \leq \int_\Omega |\nabla v|^2 dx + cap_\nu(T) \int_\Gamma v^2 d\sigma.$$

for all $v \in \mathcal{C}_0^\infty(\Omega)$ and $v \geq 0$.

Sketch of Proof.

1. For any given v, let $v_\varepsilon = (w_\varepsilon - 1)(\psi - v)^+ + (\psi - v)^- + \psi$. Then $v_\varepsilon \in K_\varepsilon = \{v \in H_0^1(\Omega); v \geq \psi_\varepsilon\}$.
2. Since w_ε converges to 0 weakly in $H_0^1(\Omega)$, $v_\varepsilon \to v$ weakly in $H_0^1(\Omega)$.
3. From the minimality of u_ε, $J(u_\varepsilon) \leq J(v_\varepsilon)$.
4. Finally, by using the proposition, we have

$$\limsup J(v_\varepsilon) \leq J_\nu(v).$$

Proof of Theorem

Proof of Main Theorem.

1. Since ψ^+ is in the set K_ε for all ε, $J(u_\varepsilon) \le J(\psi^+)$ and hence $\|u_\varepsilon\|_{H^1}$ is bounded uniformly on ε. From this, we can extract a subsequence u_{ε_j} which converges to u weakly in H^1.
2. From previous two lemmas,

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + fudx + cap_\nu(T)\frac{1}{2}\int_\Gamma((\psi-u)^+)^2 d\sigma$$
$$\le \liminf_{\varepsilon\to 0}\int_\Omega \frac{1}{2}|\nabla u_\varepsilon|^2 + fu_\varepsilon dx \le \limsup_{\varepsilon\to 0}\int_\Omega \frac{1}{2}|\nabla u_\varepsilon|^2 + fu_\varepsilon dx$$
$$\le \int_\Omega \frac{1}{2}|\nabla v|^2 + fvdx + cap_\nu(T)\int_\Gamma((\psi-v)^+)^2 d\sigma,$$

for all $v \in \mathcal{C}_0^\infty(\Omega)$ with $v \ge 0$.

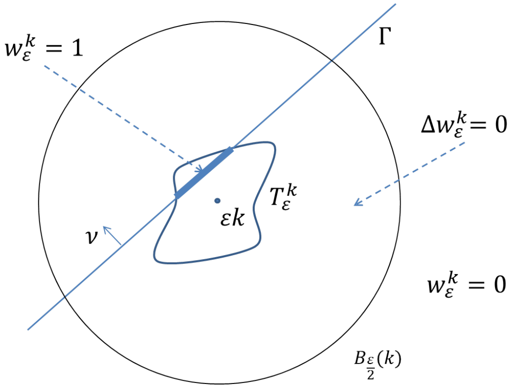
□

Construction of Correctors

We are going to construct w_ε in proposition by defining w_ε locally near the intersection between Γ and T_ε^k.
Let $\gamma_\varepsilon^k = \Gamma \cap T_\varepsilon^k$ and let w_ε^k be the restriction of w_ε to $Q_\varepsilon(\varepsilon k)$ given by

$$\begin{cases} w_\varepsilon^k = 1 & \text{on } \gamma_\varepsilon^k \\ \Delta w_\varepsilon^k = 0 & \text{in } B_{\varepsilon/2} \setminus \gamma_\varepsilon^k \\ w_\varepsilon^k = 0 & \text{in } Q_\varepsilon \setminus B_{\varepsilon/2}, \end{cases}$$

Construction of Correctors

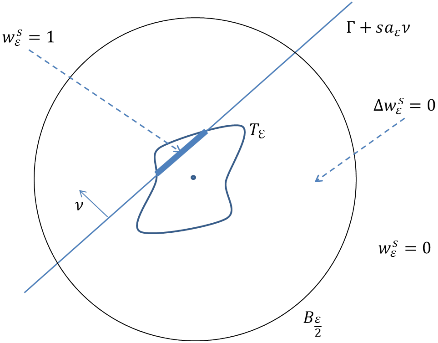


If we define $w_\varepsilon = \sum_k w_\varepsilon^k$, the energy of w_ε is given by

$$\int_\Omega |\nabla w_\varepsilon|^2 dx = \sum_k \int_\Omega |\nabla w_\varepsilon^k|^2 dx.$$

Construction of Correctors

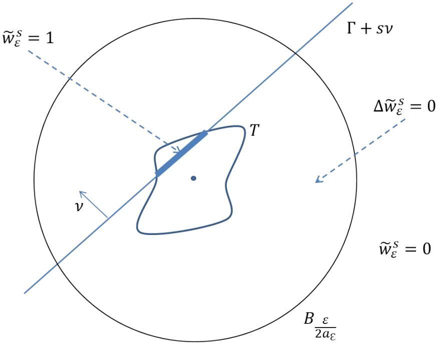
So, we need to calculate the energy $\int_\Omega |\nabla w_\varepsilon^k|^2 dx$.
To make our problem more simple, we assume $\Gamma = \{x \cdot \nu = 0\}$.
Let $g_\varepsilon(s) = \int_{B_{\varepsilon/2}} |\nabla w_\varepsilon^s|^2 dx$ where w_ε^s solves the local corrector equation defined as follow



Construction of Correctors

Define $\widetilde{w}_\varepsilon^s = w_\varepsilon^s(\frac{x}{a_\varepsilon})$ and $G_\varepsilon(s) = \int_{B_{\varepsilon/a_\varepsilon}} |\nabla \widetilde{w}_\varepsilon^s|^2 dx$. Then, we have

$$g_\varepsilon(s) = a_\varepsilon^{n-2} G_\varepsilon(s)$$



Construction of Correctors

Lemma

$G_\varepsilon(s)$ converges to $f(s) = cap\,(T \cap \Gamma_\nu(s))$ as $\varepsilon \to 0$ where $\Gamma_\nu(s) = \Gamma + (s)\nu$. Moreover, the convergence is uniform on s.

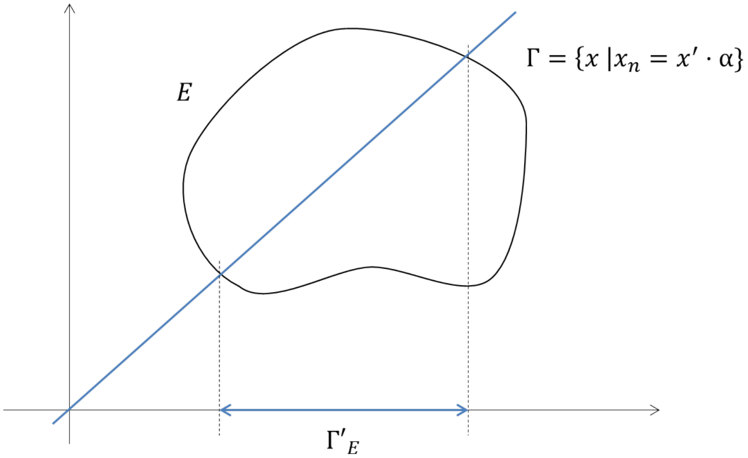
From now on, we assume $\nu_n \neq 0$.
We define

$$\Gamma'_E = \mathrm{proj}_{n-1}(E \cap \Gamma), \quad \mathcal{Z}_\varepsilon = \varepsilon^{-1}\Gamma'_E \cap \mathbb{Z}^{n-1}.$$

Then Γ may be represented as

$$\Gamma \cap E = \{(x', \alpha \cdot x'); x' \in \Gamma'_E\}, \quad \alpha = (-\nu_1/\nu_n, \cdots, -\nu_{n-1}/\nu_n)$$

Construction of Correctors



Note that $\sigma(\Gamma'_E) = \nu_n \sigma(E \cap \Gamma)$ where σ is the surface measure induced by $\mathbb{R}^n$.

Construction of Correctors

Lemma

For any measurable subset E *of* $\mathbb{R}^n$ *and for a.e.* $\nu \in S^{n-1}$,

$$\int_E |\nabla w_\varepsilon|^2 dx \to \sigma(\Gamma \cap E)\, cap_\nu(T).$$

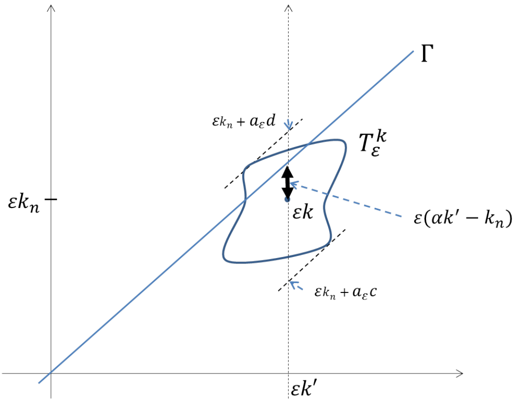
if we choose $a_\varepsilon = \varepsilon^{\frac{n}{n-1}}$.

Construction of Correctors

Proof.

Note that $(\varepsilon k + T_\varepsilon) \cap \Gamma \neq \emptyset$ $(k = (k', k_n))$ is equivalent to the condition

$$\varepsilon \left(\alpha \cdot k' - k_n\right) \in (a_\varepsilon c, a_\varepsilon d).$$



Construction of Correctors

proof continued.

Let $t = t(k') = \varepsilon\,(\alpha \cdot k' - k_n)$. Then $t \in \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$.
And, since

$$-\varepsilon k + ((\varepsilon k + T_\varepsilon) \cap \Gamma) = T_\varepsilon \cap (t(k')e_n + \Gamma)\,,$$

the shape of $((\varepsilon k + T_\varepsilon) \cap \Gamma)$ is completely determined by $t = t(k')$. Hence we have

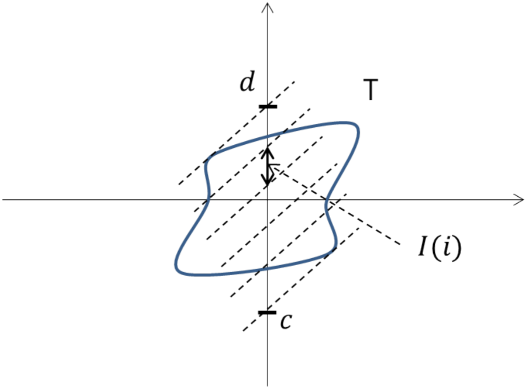
$$\int_{B_{\varepsilon/2}(\varepsilon k)} |\nabla w_\varepsilon^k|^2 dx = \int_{B_{\varepsilon/2}} |\nabla w_\varepsilon^{(t/a_\varepsilon)\nu_n}|^2 dx = g_\varepsilon((t/a_\varepsilon)\nu_n).$$

□

Construction of Correctors

proof continued.

For given any $M \in \mathbb{N}$, let $\delta = \frac{d-c}{M}$ and $I(i)$ defined as follow:



□

Construction of Correctors

proof continued.

Since $\cup_{i=1}^{M} I(i) = (c,d)$, we have

$$\int_E |\nabla w_\varepsilon|^2 dx = \sum_{i=1}^{M} \sum_{t(k')/a_\varepsilon \in I(i)} \int_{B_{\varepsilon/2}(\varepsilon k)} |\nabla w_\varepsilon^k|^2 dx$$

Let $N_\varepsilon = \#\mathcal{Z}_\varepsilon$. And $A_i(\varepsilon) = \#\{t(k')/a_\varepsilon \in I(i) : k' \in \mathcal{Z}_\varepsilon\}$.
Note that $t(k')/\varepsilon$ are distributed in a unit interval.
We assume that those points distributed uniformly. In other words, we have

$$A_i(\varepsilon) = (1+\rho(\varepsilon))N(\varepsilon)\frac{a_\varepsilon \delta}{\varepsilon}, \quad \rho(0+) = 0.$$

□

Construction of Correctors

proof continued.

From those, we have

$$\begin{aligned}\int_E |\nabla w_\varepsilon|^2 dx &= \sum_{i=1}^{M} \sum_{t(k')/a_\varepsilon \in I(i)} \int_{B_{\varepsilon/2}(\varepsilon k)} |\nabla w_\varepsilon^k|^2 dx \\ &\leq \sum_{i=1}^{M} A_i(\varepsilon) \sup_{t/a_\varepsilon \in I(i)} g_\varepsilon((t/a_\varepsilon)\nu_n) \\ &\leq (1+\rho(\varepsilon))N(\varepsilon)\frac{a_\varepsilon \delta}{\varepsilon} \sum_{i=1}^{M} \sup_{t/a_\varepsilon \in I(i)} g_\varepsilon((t/a_\varepsilon)\nu_n) \\ &\leq (1+\tilde{\rho}(\varepsilon))\frac{\sigma(\Gamma'_E)}{\varepsilon^{n-1}}\frac{a_\varepsilon \delta}{\varepsilon} \sum_{i=1}^{M} a_\varepsilon^{n-2} \sup_{t/a_\varepsilon \in I(i)} G_\varepsilon((t/a_\varepsilon)\nu_n),\end{aligned}$$

where $\tilde{\rho}(0+) = 0$. □

Construction of Correctors

proof continued.

Taking limit superior on both sides we obtain, by the uniform convergence of G_ε,

$$\limsup_{\varepsilon\to 0}\int_E |\nabla w_\varepsilon|^2 dx \le \sigma(\Gamma'_E)\frac{(d-c)}{M}\sum_{i=1}^{M}\sup_{\tilde{t}\in\tilde{I}(i)} f(\tilde{t}\nu_n).$$

Then, passing to the limit $M\to\infty$,

$$\limsup_{\varepsilon\to 0}\int_E |\nabla w_\varepsilon|^2 dx \le \sigma(\Gamma'_E)\int f(\tilde{t}\nu_n)dt$$

$$= \frac{\sigma(\Gamma'_E)}{\nu_n}\int f(s)ds = \sigma(\Gamma\cap E)\int f(s)ds.$$

In a completely analogous way, we find

$$\liminf_{\varepsilon\to 0}\int_E |\nabla w_\varepsilon|^2 dx \ge \sigma(\Gamma\cap E)\int f(s)ds.$$

Proof of Proposition

Proposition (Existence of Corrector)

Let $\Gamma = \{x \in \mathbb{R}^n ; (x - x_0) \cdot \nu = 0\}$ *and let* $a_\varepsilon = \varepsilon^{\frac{n}{n-1}}$. *Then, for almost all* $\nu \in S^{n-1}$, *there exists a sequence of functions* w_ε *satisfying*

1. $w_\varepsilon = 1$ *on* Γ_ε
2. $w_\varepsilon \to 0$ *weakly in* $H_0^1(\Omega)$
3. *For any sequence* $z_\varepsilon \in H_0^1(\Omega)$ *such that* $z_\varepsilon = 0$ *on* Γ_ε, $\|z_\varepsilon\|_{L^\infty(\Omega)} \leq C$ *and* $z_\varepsilon \to z$ *weakly in* $H_0^1(\Omega)$ *there holds*

$$\lim_{\varepsilon \to 0} < \triangle w_\varepsilon, \rho z_\varepsilon >_{H^{-1}, H_0^1} \to cap_\nu(T) \int_\Gamma \rho z d\sigma \tag{3.2}$$

for all $\rho \in \mathcal{C}_0^\infty(\Omega)$.

Proof of Proposition

Lemma (Compact embedding with $o(\varepsilon)$ error)

Suppose $v_\varepsilon \to v$ *weakly in* $H_0^1(\Omega)$ *and let* $\Gamma = \Omega \cap \{x_n = 0\}$. *Then*

$$\frac{1}{\varepsilon}\int_0^\varepsilon \int_\Gamma (v_\varepsilon(x', x_n) - v(x', 0))dx'dx_n \to 0.$$

Next we note that there exist measures μ_ε^k and ν_ε^k such that

$$\Delta w_\varepsilon^k = \mu_\varepsilon^k - \nu_\varepsilon^k, \quad \operatorname{supp} \mu_\varepsilon^k \subset \partial B_\varepsilon(\varepsilon k), \ \operatorname{supp} \nu_\varepsilon^k \subset \gamma_\varepsilon^k.$$

We define

$$\mu_\varepsilon = \sum_k \mu_\varepsilon^k.$$

Proof of Proposition

Lemma

For a.e. $\nu \in S^{n-1}$, $\mu_\varepsilon \to cap_\nu(T)\sigma$ *in the weakly star sense of measures. That is,*

$$\langle \mu_\varepsilon, \rho \rangle \to cap_\nu(T) \int_\Gamma \rho d\sigma, \text{ for all } \rho \in C_c^\infty(\Omega).$$

Proof.

Since $(1 - w_\varepsilon)$ is zero on Γ_ε and 1 on $\cup_k \partial B_\varepsilon(\varepsilon k)$ we get

$$\int_\Omega d\mu_\varepsilon = \int_\Omega \Delta w_\varepsilon (1 - w_\varepsilon) = \int_\Omega |\nabla w_\varepsilon|^2 dx \leq C.$$

Thus $\mu_\varepsilon \to \mu$ in the weak-* sense of measure for some finite measure μ.
Clearly, $\operatorname{supp} \mu \subset \Gamma$ and,

$$\int_E d\mu = \lim_\varepsilon \int_E d\mu_\varepsilon = \int_E |\nabla w_\varepsilon|^2 dx \to cap_\nu(T)\sigma(E \cap \Gamma)$$

tells us the shape of μ.

Proof of Proposition

Let $v_\varepsilon(x', x_n) = (z_\varepsilon \rho)(x', \varepsilon x_n)$ and $v(x', x_n) = (z\rho)(x', 0)$. Then, from the compact embedding lemma, we have

$$\frac{1}{\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} \int_\Pi |(z_\varepsilon \rho)(x', x_n) - (z\rho)(x', 0)| dx' dx_n$$
$$= \int_{-1/2}^{1/2} \int_\Pi |(z_\varepsilon \rho)(x', \varepsilon x_n) - (z\rho)(x', 0)| dx' dx_n \to 0$$

and thus

$$(z_\varepsilon \phi)(x', \varepsilon x_n) =: v_\varepsilon(x', x_n) \to v(x', x_n) := (z\rho)(x', 0),$$

a.e. on $S := \Gamma \times (-1/2, 1/2)$.

Proof of Proposition

By Egoroff's theorem we can assert the existence of a set S_δ such that

$$v_\varepsilon \to v \text{ uniformly on } S_\delta, \quad |S \setminus S_\delta| < \delta,$$

for any $\delta > 0$.
Upon rescaling we find: There exists $\varepsilon_0 > 0$ such that

$$|(z_\varepsilon \rho)(x', x_n) - (z\rho)(x', 0)| < \delta \text{ on } S_\delta^\varepsilon, \quad |S^\varepsilon \setminus S_\delta^\varepsilon| < \varepsilon\delta, \quad \text{for all } \varepsilon < \varepsilon_0,$$

where, for any set $E \in \mathbb{R}^n$, $E^\varepsilon = \{(x', \varepsilon x_n) : (x', x_n) \in E\}$.
Note that since $\Delta w_\varepsilon = \mu_\varepsilon - \nu_\varepsilon$ with $\operatorname{supp} \nu_\varepsilon \subset \Gamma_\varepsilon$ and $z_\varepsilon = 0$ on Γ_ε,

$$\int_\Omega \Delta w_\varepsilon \rho z_\varepsilon dx = \int_\Omega \rho z_\varepsilon d\mu_\varepsilon.$$

This allows us to compute

$$\left| \int_\Omega (z_\varepsilon \rho - z\rho) d\mu_\varepsilon \right| \leq \int_{S_\delta^\varepsilon} |z_\varepsilon \rho - z\rho| d\mu_\varepsilon + 2\|z\rho\|_{L^\infty} \int_{S^\varepsilon \setminus S_\delta^\varepsilon} d\mu_\varepsilon.$$

Proof of Proposition

The first term converges to 0 because $z_\varepsilon \phi$ converges to $z\phi$ uniformly on S_δ^ε and the last term is smaller than $C\delta$ for arbitrary $\delta > 0$. So, $\left| \int_\Omega (z_\varepsilon \rho - z\rho) d\mu_\varepsilon \right|$ should be converge to 0.

Finally, we are done if we prove

$$\lim_{\varepsilon \to 0} \int_\Omega z\rho(x', 0) d\mu_\varepsilon = cap_\nu(T) \int_\Gamma z\rho d\sigma.$$

Uniform Distribution for the 1-dimensional sequence

Definition (Uniform distribution mod 1)

1. Let $\{x_j\}_{j=1}^{\infty}$ be given sequence of real numbers. For a positive integer N and a subset E of $[0,1]$, let the counting function $A\left(E;\{x_j\};N\right)$ be defined as the numbers of terms x_j, $1 \leq j \leq N$, for which $x_j \in E$ (mod 1).
2. The sequence of real numbers $\{x_j\}$ is said to be uniformly distributed modulo 1 if for every pair a, b of real numbers with $0 \leq a < b \leq 1$ we have

$$\lim_{N\to\infty} \frac{A\left([a,b);\{x_j\};N\right)}{N} = b - a.$$

Uniform Distribution for the 1-dimensional sequence

Lemma (Weyl's criterion)

A sequence $\{x_j\}_{j=1}^{\infty}$ *is uniformly distributed mod 1 if and only if*

$$\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i l x_j} \to 0, \text{ as } N \to \infty,$$

for any nonzero $l \in \mathbb{Z}$*.*

From the Weyl's criterion, we can easily check that the sequence $\{k\alpha\}_{k\in\mathbb{N}}$ is uniformly distributed modulo 1 sense if α is a irrational number.
In other words, the sequence $\{k\alpha\}$ satisfies the following:

$$A([a,b);\{x_j\},N) = (b-a)N(1+\rho(N^{-1})),$$

Uniform Distribution for the 1-dimensional sequence

But, we cannot assert that above is true when the length of interval is shrinking as $j \to \infty$. This is a much stronger result that relies on deeper arithmetic properties of α.

Definition

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of real numbers. The discrepancy of its N first elements is the number

$$D_N(\{x_j\}_{j=1}^{N}) = \sup_{0 \leq a < b \leq 1} \left| \frac{A([a,b); \{x_j\}_{j=1}^{\infty}, N)}{N} - (b-a) \right|.$$

If $x_j = j\alpha$, we simply write $D_N(\alpha)$.

Uniform Distribution for the 1-dimensional sequence

Theorem (Kesten,1964)

$$\frac{N D_N(\alpha)}{\log N \log(\log N)} \to \frac{2}{\pi^2}$$

in measure w.r.t. α *as* $N \to \infty$. *In particular, this result is true for a.e.* α *in a bounded set.*

Corollary

For a.e. $\alpha \in \mathbb{R}$ *holds*

$$D_N(\alpha) = O\left(\frac{\log^{2+\delta} N}{N}\right),$$

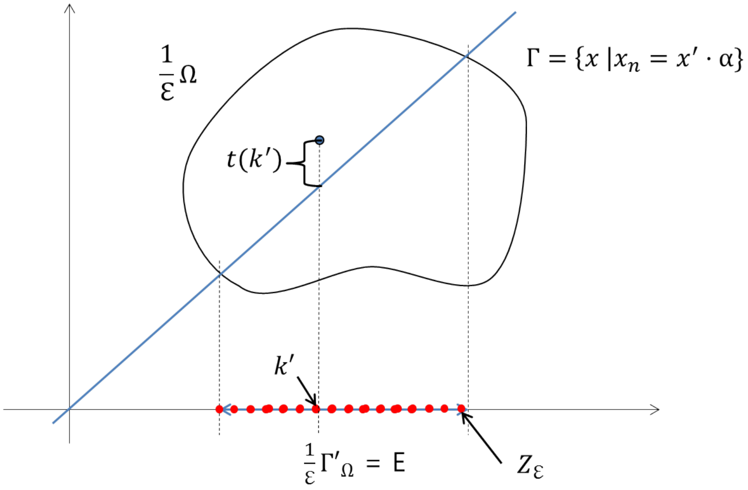
for any $\delta > 0$.

Definition

If $\alpha \in \mathbb{R}$ satisfies the condition above, then we write

$$\alpha \in \mathcal{A}.$$

Application to the Sequence $\{k'\alpha\}$



Application to the Sequence $\{k'\alpha\}$

Lemma

Let $\alpha = (\alpha_1, \cdots, \alpha_m)$ *be any vector in* $\mathbb{R}^m$. *Suppose that* $E \subset \mathbb{R}^m$ *is a (regular) subset with positive measure. Suppose also that* $\alpha_i \in \mathcal{A}$, *for at least one* $i \in \{1, \cdots, m\}$.
Let

$$N(\varepsilon) = \#\left(E \cap \mathbb{Z}^m\right)$$
$$A(\varepsilon^p, c) := \#\{k' \in E \cap \mathbb{Z}^m : t(k')/\mathbb{Z} =\in (c, c + \varepsilon^p)/\mathbb{Z}\}$$

Then for any $0 < p < 1$,

$$A(\varepsilon^p, c) = (1 + \rho(\varepsilon))N(\varepsilon)\varepsilon^p, \text{ for some } \rho \text{ such that } \rho(0^+) = 0.$$

Application to the Sequence $\{k'\alpha\}$

Proof.

Suppose first that E is a cube, $E = x + (a,b)^m$. Without loss of generality, we may assume $\alpha_m \in \mathcal{A}$.
Let

$$S'_\varepsilon := \{k' \in \mathbb{Z}^{m-1} : (k', k_m) \in \varepsilon^{-1}E, \text{ for some } k_m \in \mathbb{Z}\}.$$

If $k' \in S'_\varepsilon$, there exists integers m_ε and M_ε such that

$$(k', k_m) \in (\varepsilon^{-1}E) \cap \mathbb{Z}^m, \quad \text{for } m_\varepsilon \le k_m \le M_\varepsilon.$$

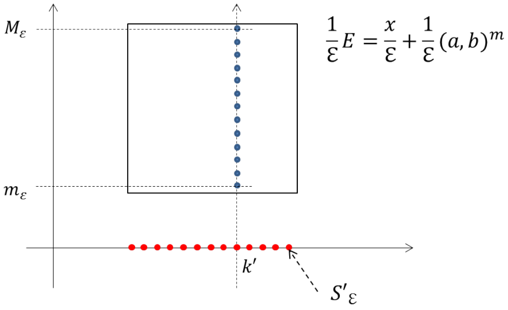
Hence we have

$$N(\varepsilon) = (\#S'_\varepsilon)H_\varepsilon, \quad H_\varepsilon = M_\varepsilon - m_\varepsilon.$$

□

Application to the Sequence $\{k'\alpha\}$

proof continued.



Application to the Sequence $\{k'\alpha\}$

proof continued.

Let

$$A_{k'}(\varepsilon^p, c) = \#\{k_m : \alpha' \cdot k' + \alpha^m k_m \in (c, c+\varepsilon^p)/\mathbb{Z}, m_\varepsilon \leq k_m \leq M_\varepsilon\},$$

Then,

$$A(\varepsilon^p, c) = \sum_{k' \in S'_\varepsilon} A_{k'}(\varepsilon^p, c).$$

From the definition of $A(\varepsilon^p, t)$, we conclude that

$$\begin{aligned} A_{k'}(\varepsilon^p, c) &= \#\{k_m : \alpha' \cdot k' + \alpha_m k_m \in (c, c+\varepsilon^p)/\mathbb{Z}, m_\varepsilon \leq k_m \leq M_\varepsilon\} \\ &= \#\{k_m : k_m \alpha_m \in (\tilde{c}, \tilde{c}+\varepsilon^p)/\mathbb{Z},\ 1 \leq k_m \leq H_\varepsilon + 1\} \end{aligned}$$

for some $\tilde{c}$. □

Application to the Sequence $\{k'\alpha\}$

proof continued.

By applying Kesten's theorem to the set $\{k_m : k_m\alpha_m \in [\tilde{c}, \tilde{c}+\varepsilon^p]/\mathbb{Z},\ 1 \leq k_m \leq H_\varepsilon + 1\}$, we have

$$\left| \frac{A_{k'}(\varepsilon^p, c))}{H_\varepsilon} - |(\tilde{c}, \tilde{c}+\varepsilon^p)| \right| = \left| \frac{A_{k'}(\varepsilon^p, c)}{H_\varepsilon} - \varepsilon^p \right| \leq D_{H_\varepsilon}(\alpha_m) = o\left(H_\varepsilon^{-p}\right), \quad 0 < p < 1.$$

And hence

$$\left| \frac{A(\varepsilon^p, c)}{N(\varepsilon)} - \varepsilon^p \right| \leq \frac{H_\varepsilon}{N_\varepsilon} \sum_{k' \in S'_\varepsilon} \left| \frac{A_{k'}(\varepsilon^p, c)}{H_\varepsilon} - \varepsilon^p \right| \leq \frac{H_\varepsilon}{N_\varepsilon} \sum_{k' \in S'_\varepsilon} D_{H_\varepsilon}(\alpha^m) = D_{H_\varepsilon}(\alpha_m) = o(\varepsilon^p), \text{ as } \varepsilon \to 0.$$

□

Application to the Sequence $\{k'\alpha\}$

proof continued.

It can be easily extended when E is a finite union of cubes.
Finally, when E is just measurable, approximate E by a finite union of cubes U.
Then, by applying the previous result, we get the conclusion. □

Thank You!